The homotopy Lie algebra of a complex hyperplane arrangement is not necessarily finitely presented.

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Abstract.

We present a theory that produces several examples where the homotopy Lie algebra of a complex hyperplane arrangement is not finitely presented. We also present examples of hyperplane arrangements where the enveloping algebra of this Lie algebra has an irrational Hilbert series. This answers two questions of Denham and Suciu.

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0. Introduction

Let $\mathcal{A} = \{H\}$ be a finite set of complex hyperplanes in $\mathbf{C}^n$, i.e. a *complex hyperplane arrangement* in $\mathbf{C}^n$ and let X be the complement of their union in $\mathbf{C}^n$:

$$X = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

The cohomology of X is called the Orlik-Solomon algebra and the Yoneda Ext-algebra of $H^*(X)$ is a Hopf algebra which is the enveloping algebra of a graded Lie algebra, which is called the homotopy Lie algebra of the arrangement $\mathcal{A}$. In this paper we calculate explicitly this Lie algebra in several cases and in particular we show by explicit examples that this Lie algebra is not necessarily finitely presented and not even finitely generated. Furthermore we present examples of hyperplane arrangements where the enveloping algebra of this Lie algebra has an irrational Hilbert series. This solves two open problems from [De-Su], Question 1.7 p. 321. We also have some results about how often these two phenomena occur.

1. An explicit example

It is useful to begin with an explicit example. Let $\mathcal{A} = \{x, y, z, x+y, x+z, y+z\}$ be the well-known complex hyperplane arrangement, which is the smallest formal arrangement whose Orlik-Solomon algebra is nonquadratic (cf. [Sh-Yu], p. 487, Example 5.1). We know that the Orlik-Solomon algebra of $\mathcal{A}$ is the quotient of the exterior algebra in 6 variables $e_1, e_2, e_3, e_4, e_5, e_6$ by the twosided ideal generated by the four elements:

$$(e_2 - e_6)(e_3 - e_6), \quad (e_1 - e_3)(e_3 - e_5), \quad (e_1 - e_4)(e_2 - e_4), \quad (e_3 - e_4)(e_4 - e_6)(e_5 - e_6)$$

Thus if we introduce new variables $x_1, x_2, x_3, x_4.x_5, z$ by $x_i = e_i - e_6$ for $1 \le i \le 5$ and $z = e_6$ our Orlik-Solomon algebra can be written (we use that $e_1 - e_4 = (e_1 - e_6) - (e_4 - e_6) = x_1 - x_4$ etc.) as a quotient of an exterior algebra:

$$OS_{\mathcal{A}} = \frac{E(x_1, x_2, x_3, x_4, x_5, z)}{(x_2x_3, (x_1 - x_3)(x_3 - x_5), (x_1 - x_4)(x_2 - x_4), (x_3 - x_4)x_4x_5)}$$

where z does not occur among the relations. Therefore the Orlik-Solomon algebra decomposes into a tensor product of algebras (all algebras are considered over a field k of characteristic 0):

$$OS_{\mathcal{A}} = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_3 - x_1x_5 + x_3x_5, x_1x_2 - x_1x_4 + x_2x_4, x_3x_4x_5)} \otimes_k E(z)$$

where $E(z)$ is the exterior algebra in one variable and where we have used that $x_i^2 = 0$. Thus the Yoneda Ext-algebra of the Orlik-Solomon algebra is the tensor product of the Ext-algebra $Ext_R^*(k, k)$ of

$$R = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_3 - x_1x_5 + x_3x_5, x_1x_2 - x_1x_4 + x_2x_4, x_3x_4x_5)} \tag{1.1}$$

and the Ext-algebra $Ext_{E(z)}^*(k, k) = k[Z]$ where the last algebra is the polynomial algebra in one variable Z, dual to z. The last algebra is "innocent" and it therefore follows that the Yoneda Ext-algebra of the Orlik-Solomon algebra is finitely presented if and only if $Ext_R^*(k, k)$ is so, where R is given by (1.1). But the following automorphism of R

$$x_1 \to x_1 + x_3, x_2 \to -x_2 + x_3, x_3 \to x_3, x_4 \to -x_2 + x_3 + x_4, x_5 \to x_5 + x_3$$

transforms R into the isomorphic algebra

$$\frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_5, (x_1 + x_2)x_4, x_3x_4x_5)} \tag{1.2}$$

which we will still denote by R. But this algebra (1.2) can now be easily analyzed: It is the "trivial extension" of a Koszul algebra

$$S = \frac{E(x_1, x_2, x_3, x_5)}{(x_2x_3, x_1x_5)} \tag{1.3}$$

by the following cyclic module M over S :

$$M = \frac{S}{(x_1 + x_2, x_3x_5)}$$

Recall that the trivial extension of any ring Λ by any two-sided Λ-module N is denoted by $\Lambda \propto N$ and consists of the pairs (λ, n) with $\lambda \in \Lambda$ and $n \in N$ with pairwise addition and multiplication

$$(\lambda, n).(\lambda', n') = (\lambda.\lambda', \lambda n' + n\lambda')$$

The Ext-algebra of $R = S \propto M$ can now be analyzed (cf. e.g. [Lö2,Theorem 3, p. 310-311]): We have a split extension of Hopf algebras

$$k \to T(s^{-1}Ext^*_S(M, k)) \to Ext^*_R(k, k) \to Ext^*_S(k, k) \to k \tag{1.4}$$

Here S is the Koszul algebra (1.3) and

$$Ext^*_S(k, k) = k < X_1, X_5 > \otimes_k k < X_2, X_3 >$$

is the tensor product of two free algebras in the dual variables X_1, X_5 and X_2, X_3 respectively, and therefore it has global dimension 2. Furthermore

$$s^{-1}Ext^*_S(M, k) = Ext^{*-1}_S(M, k)$$

and $T(s^{-1}Ext^*_S(M, k))$ is the free algebra on the graded vector space (for the *-grading in Ext) $s^{-1}Ext^*_S(M, k)$ and has global dimension 1. The spectral sequence of extensions of Hopf algebras (1.4) [Ro2]

$$E^2_{p,q} = Tor_p^{Ext^*_S(k,k)}(k, Tor_q^{T(s^{-1}Ext^*_S(M,k))}(k, k) \Rightarrow Tor_n^{Ext^*_R(k,k)}(k, k) \,(= H_n) \tag{1.5}$$

shows immediately that $Ext^*_R(k, k)$ har global dimension 3. Furthermore (1.5) degenerates into a long exact sequence:

$$0 \to E^2_{2,1} \to H_3 \to E^2_{3,0} \to E^2_{1,1} \to H_2 \to E^2_{2,0} \to E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0 \tag{1.6}$$

where the natural maps $H_i \longrightarrow E^2_{i,0}$ are *onto*. Indeed, the natural ring projection map $S \propto M \longrightarrow S$ is split by the natural ring inclusion $S \longrightarrow S \propto M$ and this leads to a splitting on the Ext-algebra level. Thus we have exact sequences

$$0 \longrightarrow E^2_{i-1,1} \longrightarrow H_i \longrightarrow E^2_{i,0} \longrightarrow 0.$$

Now for any graded connected algebra A over k, $Tor^A_1(k,k)$ measures the minimal number of generators of A, $Tor^A_2(k,k)$ measures the minimal number of relations between these generators, $Tor^A_3(k,k)$ the minimal number of relations between these relations etc., cf. the Chapter 1 ("Présentations d'algèbres connexes") of Lemaire [Lem1]. Therefore H_1 in (1.6) measures the minimal number of generators of the Ext-algebra $Ext^*_R(k,k)$ and therefore H_1 is finite-dimensional if and only if the Ext-algebra $Ext^*_R(k,k)$ is finitely generated. Similarly H_2 measures the minimal number of relations in a minimal presentation of $Ext^*_R(k,k)$ and H_3 measures the minimal number of relations between these relations. Since the $E^2_{i,0}$ are all finite-dimensional we are led to the study of

$$E^2_{i,1} = Tor_i^{Ext^*_S(k,k)}(k, s^{-1}Ext^*_S(M,k)), \text{ for } i \geq 1 \tag{1.7}$$

where the left $Ext^*_S(k,k)$-module structure of $s^{-1}Ext^*_S(M,k)$ is given by the Yoneda product (*cf.* again Theorem 3, p. 310-311 of Löfwall [Lö2]). Note that underlying our spectral sequence is the Hochschild-Serre spectral sequence and that we are in the skew-commutative setting, whereas [Lö2] is in the commutative case, but similar (easier) proofs work here in our case. Thus to show that $Ext^*_R(k,k)$ is not finitely generated we have to show that H_1 is infinite-dimensional, i.e. that $E^2_{0,1}$ is so, i.e. (cf. (1.7)) that $s^{-1}Ext^*_S(M,k)$ needs an infinite number of generators as an $Ext^*_S(k,k)$-module, i.e. we have to study the S-resolutions of $M = S/(x_1+x_2, x_3x_5)$. We also need the extra grading on R and S so that we should indeed write $R = S \propto s^{-1}M$. Now we denote the S-ideal (x_1+x_2, x_3x_5) by I so that $M = S/I$. First we observe that if we apply the functor $Ext^*_S(.,k)$ to the exact sequence of graded left S-modules

$$0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0 \tag{1.8}$$

we obtain the isomorphisms of left $Ext^*_S(k,k)$-modules:

$$Ext_S^{*-1,t}(I,k) \xrightarrow{\sim} Ext_S^{*,t}(S/I,k), \text{ for } * \geq 1 \tag{1.9}$$

where we also have inserted the inner grading t that comes from the fact that (1.8) is an exact sequence of graded modules. Note that S is a Koszul algebra, so that only the

$Ext^{i,i}_S(k,k)$ are different from zero and we still denote them by $Ext^i_S(k,k)$. Next we note that the two ideals $I_1 = (x_1+x_2)$ and $I_2 = (x_3x_5)$ in S have zero intersection. Therefore $I = I_1 \oplus I_2$ and the $Ext^*_S(k,k)$-module to the left in (1.9) decomposes into a direct sum of $Ext^*_S(k,k)$-modules:

$$Ext_S^{*-1,t}((x_1+x_2),k) \oplus Ext_S^{*-1,t}((x_3x_5),k) \tag{1.10}$$

But x_3x_5 is in the socle of S and therefore we have as graded S-modules that $(x_3x_5) \xrightarrow{\sim} s^{-2}k$ so that the right summand of (1.10) is isomorphic to $Ext_S^{*-1,t}(s^{-2}k,k)$, i.e. to $Ext_S^{*-1,t-2}(k,k)$. It remains to analyze the left summand of (1.10). But it is easy to see that $Ann_S((x_1+x_2)) = I$ so that the graded sequence of S-modules

$$0 \longrightarrow s^{-1}I \longrightarrow s^{-1}S \xrightarrow{(x_1+x_2)} S \tag{1.11}$$

where we multiply to the right with x_1+x_2 is exact. Therefore we have a short exact sequence of graded left S-modules:

$$0 \longrightarrow s^{-1}I \longrightarrow s^{-1}S \longrightarrow (x_1+x_2) \longrightarrow 0 \tag{1.12}$$

leading to the isomorphism of left $Ext^*_S(k,k)$-modules:

$$Ext_S^{*-1,t}(s^{-1}I,k) \xrightarrow{\sim} Ext_S^{*,t}((x_1+x_2),k), \text{ for } * \geq 1 \tag{1.13}$$

and using (1.9) once more we obtain that

$$Ext_S^{*-1,t}(s^{-1}I,k) = Ext_S^{*-1,t-1}(I,k) \xrightarrow{\sim} Ext_S^{*,t-1}(S/I,k) \tag{1.14}$$

leading to the final isomorphism (combining (1.9),(1.10),(1.13),(1.14)):

$$Ext_S^{*,t}(S/I,k) \xrightarrow{\sim} Ext_S^{*-1,t-1}(S/I,k) \oplus Ext_S^{*-1,t-2}(k,k) \tag{1.15}$$

for $* \geq 1$, where the summand $Ext_S^{*-1,t-2}(k,k)$ is non-zero only if $* = t-1$. This proves everything since we see, using (1.15) that $Ext_S^{*,t}(S/I,k)$ needs a new $Ext^*_S(k,k)$-generator for $* = t-1$ for each $t = 2,3,4,\ldots$.

In particular, if we introduce for any graded module N over a graded k-algebra G the double series

$$P_G^N(x,y) = \sum_{i\geq 0, j\geq 0} |Ext_G^{i,j}(N,k)| x^i y^j \tag{1.16}$$

(where as always for a k-vector space V we denote by $|V|$ its dimension) and if we denote $P_G^k(x,y)$ by $P_G(x,y)$, we then deduce from (1.15) and the fact that $P_S(x,y) = 1/(1-2xy)^2$ that

$$P_S^{S/I}(x,y) = \frac{1}{1-xy} + \frac{xy^2}{(1-xy)(1-2xy)^2}$$

so that

$$P_{S \propto s^{-1}S/I}(x,y) = \frac{P_S(x,y)}{1 - xyP_S^{S/I}(x,y)}$$

leading to the theorem:

THEOREM 1.1. The Orlik-Solomon algebra of the complex hyperplane arrangement $\mathcal{A} = \{x, y, z, x+y, x+z, y+z\}$ is the tensor product of the exterior algebra in one variable with an algebra R whose Yoneda Ext-algebra $Ext_R^*(k,k)$ has a bigraded generating series:

$$P_R(x,y) = \frac{P_S(x,y)}{1 - xyP_S^M(x,y)} = \frac{1-xy}{1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3} \qquad (1.17)$$

where S and M are defined above. Furthermore, the Ext-algebra $Ext_R^*(k,k)$ has global dimension 3 and it has 5 generators in degree 1 and needs one new generator in each degree ≥ 2. In particular the homotopy Lie algebra of $\mathcal{A}$ is not finitely generated.

We will come back to this result in the next section.

2. The holonomy and homotopy Lie algebra of an arrangement.

The preceding analysis of the $\mathcal{A}$ arrangement in section 1 was intended to give the "simplest possible proof" that the Ext-algebra $Ext_R^*(k,k)$ was not finitely generated. However, in order to be able to analyze more cases we need a more general theory. I will here briefly describe the basics of such a theory and apply it as an alternative to our first case and then treat another case of arrangements where we can prove that the homotopy Lie algebra is also non-finitely presented. Note that the graded algebra R of the previous section has Hilbert Series $1 + 5z + 7z^2$. Let us now start with any algebra R which is a quotient of an exterior algebra $E(x_1, x_2, \ldots, x_n)$ by a homogeneous ideal J generated by elements of deegree ≥ 2. Thus $R = E(x_1, \ldots, x_n)/J$. Let m be the ideal of R generated by $(x_1, \ldots, x_n)$, and consider the exact sequence of left R-modules:

$$0 \longrightarrow m/m^2 \longrightarrow R/m^2 \longrightarrow R/m \longrightarrow 0, \qquad (2.1)$$

Now apply the functor $Ext^*_R(,k)$ to the exact sequence (2.1). We get a long exact sequence which can be written as an exact sequence of left $Ext^*_R(k,k)$-modules (we use the Yoneda product):

(2.2)

$$0 \to s^{-1}\overline{S}_m \to s^{-1}\overline{Ext}^*_R(R/m^2,k) \to Ext^*_R(k,k) \otimes Ext^1_R(k,k) \to Ext^*_R(k,k) \to S_m \to 0$$

where S_m is defined as the image of the natural map

$$Ext^*_R(k,k) \longrightarrow Ext^*_R(R/m^2,k) \tag{2.3}$$

and where e.g. $\overline{S}_m$ means that we take the elements of S_m of degrees > 0 and where s^{-1} is "the suspension" as before. Now the Ext-algebra $B = Ext^*_R(k,k)$ is bigraded and we recall that its bigraded Hilbert series is denoted by

$$P_R(x,y) = \sum_{i,j\geq 0} |Ext^{i,j}_R(k,k)| x^i y^j = B(x,y),$$

where as always for a k-vector space we denote by $|V|$ its dimension. The subalgebra A of $Ext^*_R(k,k)$ generated by $Ext^1_R(k,k)$ is also bigraded but it is situated on the diagonal so that the corresponding bigraded Hilbert series

$$A(x,y) = A(xy,1) \stackrel{def}{=} A(xy)$$

Now take the alternating sum of the two-variable Hilbert series of (2.2). We obtain:

$$S_m(x,y) - B(x,y) + xyB(x,y)|m/m^2| - x\overline{P}^{R/m^2}_R(x,y) + x(S_m(x,y) - 1) = 0 \tag{2.4}$$

where

$$\overline{P}^{R/m^2}_R(x,y) = \sum_{i>0, j\geq i} |Ext^{i,j}_R(R/m^2,k)| x^i y^j \tag{2.5}$$

We now make three fundamental observations:

1) A is a sub Hopf algebra of B and therefore according to a result of Milnor and Moore [Mi-Mo] B is free over A. Thus $S_m = B \otimes_A k$ has bigraded Hilbert series

$$S_m(x,y) = B(x,y)/A(xy) \tag{2.6}$$

2) If $m^3=0$ we have an isomorphism of left $Ext_R^*(k,k)$-modules

$$\overline{Ext}_R^*(R/m^2,k)\simeq Ext_R^*(k,k)\otimes Ext_R^1(R/m^2,k) \tag{2.7}$$

so that

$$\overline{P}_R^{R/m^2}(x,y)=B(x,y)xy^2|m^2/m^3| \tag{2.8}$$

Therefore the equality (2.4) can be written

$$\frac{B(x,y)}{A(xy)}=B(x,y)-xyB(x,y)|m/m^2|+x^2y^2B(x,y)|m^2/m^3|-x(\frac{B(x,y)}{A(xy)}-1) \tag{2.9}$$

which is another way of writing (divide by $xB(x,y)$ and use the notation $R(z)=1+|m/m^2|z+|m^2/m^3|z^2$ for the Hilbert series of R)

$$1/B(x,y)=(1+1/x)/A(xy)-R(-xy)/x \tag{2.10}$$

which is a formula due to Löfwall [Lö1].

3) In the case when $m^3=0$ the 3 middle terms of (2.2) are free $Ext_R^*(k,k)$-modules so that $s^{-1}\overline{S}_m$ is a third syzygy of a minimal graded $Ext_R^*(k,k)$-resolution of S_m. We therefore obtain the isomorphism:

$$Tor_{i,*}^B(k,S_m)\simeq Tor_{i-3,*}^B(k,s^{-1}\overline{S}_m)=Tor_{i-3,*-1}^B(k,\overline{S}_m)\quad \text{for } i\geq 3 \tag{2.11}$$

Now apply $Tor_i^B(k,)$ to the exact sequence:

$$0\longrightarrow \overline{S}_m\longrightarrow S_m\longrightarrow k\longrightarrow 0 \tag{2.12}$$

We get a long exact sequence:
(2.13)

$$\ldots\to Tor_{n+1}^B(k,k)\to Tor_n^B(k,\overline{S}_m)\to Tor_n^B(k,S_m)\overset{\varphi_n}{\to} Tor_n^B(k,k)\to Tor_{n-1}^B(k,\overline{S}_m)\ldots$$

Furthermore, since B is A-flat

$$Tor_n^B(k,S_m)=Tor_n^B(k,B\otimes_A k)=Tor_n^A(k,k)$$

and $\varphi_n: Tor_n^A(k,k)\to Tor_n^B(k,k)$ is induced by the natural inclusion $A\to B$ which is split by a ring map in the other direction: Divide $B=Ext_R^{*,*}(k,k)$ by the twosided ideal

generated by $\oplus_{j>i>0} Ext_R^{i,j}(k,k)$. Thus the maps φ_n in (R) are monomorphisms and (2.13) splits into short exact sequences, using (2.11):

$$0 \longrightarrow Tor^A_{i,j}(k,k) \longrightarrow Tor^B_{i,j}(k,k) \longrightarrow Tor^A_{i+2,j+1}(k,k) \longrightarrow 0 \tag{2.14}$$

Now we can summarize:

THEOREM 2.1. Let R be a quotient of an exterior algebra (finite number of generators in degree 1) by a homogeneous ideal generated by elements of degree ≥ 2. Let m be the augmentation ideal of R. Assume that $m^3 = 0$. Let $B = Ext_R^*(k,k)$ be the Yoneda Ext-algebra and let A be the subalgebra of B, generated by $Ext^1_R(k,k)$. Then the exact sequences (2.14) hold. In particular

a) B is finitely generated if and only if the graded vector space $Tor^A_{3,*}(k,k)$ has finite dimension.

b) B is finitely presented if and only if the graded vector spaces $Tor^A_{3,*}(k,k)$ and $Tor^A_{4,*}(k,k)$ have finite dimension.

c) B is finitely presented and has a finite number of relations between the minimal relations if and only if the graded vector spaces $Tor^A_{3,*}(k,k)$, $Tor^A_{4,*}(k,k)$ and $Tor^A_{5,*}(k,k)$ have finite dimension.

Etc.

Note that A is the enveloping algebra of a graded Lie algebra (the holonomy Lie algebra) whose ranks are equal to the ranks of the lower central series (LCS) of the fundamental group of the hyperplane complement (cf. section 6 below). Note also that in general the Hilbert series $A(x)$ of A when $m^3 = 0$ is obtained from (2.10): replace x by x/y in that formula and put $y = 0$. This gives

$$1/P_R(x/y,y)|_{y=0} = 1/A(x)$$

We can get an alternative proof of assertion about generators of the Ext-algebra in Theorem 1.1 above, using only the formula (1.17) there. The preceding recipe gives in that case: that $A(x) = (1-x)/(1-2x)^3$. Now recall that for any graded algebra A we have the following formula for the relation between its Hilbert series $A(z)$ and the Hilbert series $Tor^A_{i,*}(k,k)(z)$ of the graded Tor: (cf. eg. Lemaire [Lem1, Appendix A2]):

$$\frac{1}{A(x)} = \sum_{i\geq 0} (-1)^i Tor^A_{i,*}(k,k)(x) \tag{2.15}$$

Since A in the case of Theorem 1.1 has global dimension 3 and 5 generators in degree 1 and 7 relations in degree 2, (2.15) gives that $Tor^A_{3,*}(k,k)(x) = x^3/(1-x)$ so that we see once more that $Tor^A_{3,i}(k,k)$ is one-dimensional for all $i \geq 3$.

Remark 2.2. If $m^3 \neq 0$ but more generally

$$\overline{Ext}^*_R(R/m^i, k) \longrightarrow \overline{Ext}^*_R(R/m^{i+1}, k) \tag{2.16}$$

is 0 for $i \geq 2$ then we have the same conclusion as in Theorem 2.1 but the proof is slightly different, since now $N = \overline{Ext}^*_R(R/m^2, k)$ is not free as a $B = Ext^*_R(k, k)$-module but it has a finite homological dimension and the corresponding $Tor_i^B(k, N) \simeq m^{i+2}/m^{i+3}$ are finite-dimensional. The condition (2.16) is sometimes, but not always satisfied if $m^4 = 0$, but in the last case one can prove the validity of the fomula (2.10) is *equivalent* to the assertion that the map (2.16) is zero for $i \geq 2$ (only the case $i = 2$ is of course important in this case). This will be used below when we study graphic arrangements. Furthermore the important formula (2.10) which we will now write as

$$1/P_R(x, y) = (1 + 1/x)/R^!(xy) - R(-xy)/x \tag{2.17}$$

is still valid under (2.16) but here the Hilbert series $R(z)$ might be a polynomial of degree > 2. Note that we have written $R^!$ (instead of A); it is the Koszul dual of R.

Remark 2.3. The formula (2.17) in Remark 2.2 is the special case is a special case ($n = 3$) of a whole family of formulae ($n \geq 3$)

$$\frac{1}{P_R(x, y)} = \frac{(1 - (-x)^{2-n})}{R^!(xy)} + R(-xy)(-x)^{2-n} \tag{2.18_n}$$

The validity of (2.18_n) is a consequence of the fact that the so-called Koszul complex $R^! \otimes_k Hom_k(R, k)$ has only non-zero homology groups in degree 0 and in degree $n - 1$. For more details about this, cf. the appendix B (Theorem B.4) by Löfwall [Lö4] to [Ro3] and also [Ro5]. We will say here that R satisfies L_n if (2.18_n) holds true. For Orlik-Solomon algebras with $m^4 = 0$ we still have the formula (2.17) since the algebra is the tensor product of an algebra with $m^3 = 0$ and an "innocent" algebra $E[z]$. In the section 5 where we study the case of graphic arrangements we will see that for any $n \geq 3$ there are examples where the condition L_n holds (namely the Orlik-Solomon algebra of the graphic arrangement corresponding to an $n + 1$-gon for $n \geq 3$), but that there are also examples where none of these conditions is satisfied (these cases can however sometimes be handled with the method of [Ro5]).

3. Some other hyperplane arrangements

Some of the cases from [Su] can be treated in the same way as in section 2. Here we just briefly describe the results for the so-called X_2-arrangement which is defined by the

polynomial $xyz(x+y)(x-z)(y-z)(x+y-2z)$. Now the Orlik-Solomon algebra can be written as a quotient of the exterior algebra in 7 variables $E(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ by the ideal generated by five elements:

$$(e_5-e_7)(e_6-e_7),(e_3-e_7)(e_4-e_7),(e_2-e_6)(e_3-e_6),(e_1-e_5)(e_3-e_5),(e_1-e_2)(e_2-e_4)$$

Now isolate e_3, i.e. introduce variables $a = e_1 - e_3, b = e_2 - e_3, c = e_4 - e_3, d = e_5 - e_3, e = e_6 - e_3, f = e_7 - e_3$. Now the relations in the Orlik-Solomon algebra do not contain e_3 and this algebra is now a tensor algebra of the quotient:

$$R = \frac{E(a,b,c,d,e,f)}{(ab-ac+bc, ad, be, cf, de-df+ef)}$$

with the exterior algebra in one variable $z = e_3$. Therefore we are lead to the analysis of the Yoneda Ext-algebra of the quotient R above whose Hilbert series is $R(t) = 1+6t+10t^2$. Furthermore, let

$$S = \frac{E(a,b,c,d,e,f)}{(ab-ac+bc, ad, be, cf)}$$

and consider the ring map:

$$S \longrightarrow S/(de-df+ef) = R \tag{3.1}$$

It is not difficult to show that (3.1) is a so-called Golod map (cf. [Le2] and the litterature cited there). One finds that $S^!(t) = (1-t)^2/(1-2t)^4$,that $R^!(t) = (1-t)^4/(1-2t)^5$ and more precisely that $R^!$ has global dimension 5 and that:

$$\sum_{i\geq 0} |Tor^{R^!}_{3,i}(k,k)|z^i = 5z^4 + \frac{6z^5}{(1-z)}\,, \tag{3.2}$$

$$\sum_{i\geq 0} |Tor^{R^!}_{4,i}(k,k)|z^i = 2z^6 + \frac{(6-z)z^7}{(1-z)^2} \tag{3.3}$$

and

$$\sum_{i\geq 0} |Tor^{R^!}_{5,i}(k,k)|z^i = \frac{z^{10}}{(1-z)^4} \tag{3.4}$$

Thus using the theory from section 2 we see that the homotopy Lie algebra of the arrangement X_2 is "extremely non-finitely presented": It needs an infinite number of generators

(3.2), and the minimal number of relations between a minimal system of generators is infinite (3.3) and the minimal number of relations between the relations is infinite (3.4). Among the graphic arrangements (cf. section 5 below for more details) there are however more arrangements with finitely presented Ext-algebra than infinitely presented ones.

We finish this section with one unsolved case: Recall that the so-called non-Fano arrangement is the hyperplane arrangement defined by $xyz(x-y)(x-z)(y-z)(x+y-z)$. In this case the corresponding R (we have eliminated one variable as above) has Hilbert series $(1+3t)^2$ but the corresponding $R^!(t)$ is rather complicated. We have however managed to calculate the LCS-ranks two steps higher than in Sucio [Su], using the Backelin et al programme BERGMAN [B]; with the notations of [Su] we have $\phi_8 = 3148$ and $\phi_9 = 9857$, but for the last result we needed 64-bits PSL on an AMD opteron machine with 12 GB of internal memory.

4. Arrangements with irrational Hilbert series.

In [De-Su] it is also asked if the enveloping algebra of the homotopy Lie algebra of an arrangement can have an irrational Hilbert series.

We will here describe one case we have found where this is conjecturally true and a second case where this is *proved* to be true. This development is rather recent: we found the second case only recently and the first (more complicated) case is the well-known Mac Lane arrangement, whose amazing homological properties we also discovered recently. The proof in the second case (the first case is probably treated in a similar but more complicated way) is based on ideas of the present paper, but involves a lot more new ideas and will be presented in another paper in preparation [Ro6]. Let us just indicate some more details:

First case (the Mac Lane arrangement):

Recall that the Mac Lane arrangement defined by the annihilation of the polynomial $Q = xyz(y-x)(z-x)(z+\omega y)(z+\omega^2 x+\omega y)(z-x-\omega^2 y)$ in $\mathbf{C}^3$ where $\omega = e^{2\pi i/3}$. It is not difficult to see that with the notations of our section 1 above the Orlik-Solomon algebra of the Mac Lane arrangement is $R \otimes E[z]$ where R is the quotient of the exterior algebra $E(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in 7 variables with the ideal generated by the 8 quadratic elements $x_1x_2 - x_1x_4 + x_2x_4, x_1x_3 - x_1x_5 + x_3x_5, x_1x_6 - x_1x_7 + x_6x_7,$ $x_2x_3 - x_2x_6 + x_3x_6, x_4x_5 - x_4x_7 + x_5x_7, x_2x_5, x_4x_6, x_3x_7$. This ring R has Hilbert series $R(z) = 1 + 7\,z + 13\,z^2$ and therefore the formula (2.17) above can be applied, and the only thing needed to be proved is that the Koszul dual $R^!$ of R has an irrational Hilbert series. But this Koszul dual $R^! = U(g)$ is the quotient of the free associative algebra $k < X_1, X_2, \dots, X_7 >$ in the seven dual variables by the two-sided ideal generated by the 13 dual relations among the Lie-commutators $[X_i, X_j] = X_iX_j - X_jX_i$ for $i \neq j$:

$[X_1, X_2] + [X_1, X_4], [X_1, X_4] + [X_2, X_4], [X_1, X_3] + [X_1, X_5], [X_1, X_5] + [X_3, X_5], [X_1, X_6] + [X_1, X_7], [X_1, X_7] + [X_6, X_7], [X_2, X_3] + [X_2, X_6], [X_2, X_6] + [X_3, X_6], [X_4, X_5] + [X_4, X_7], [X_4, X_7] + [X_5, X_7], [X_2, X_7], [X_3, X_4], [X_5, X_6]$ so that the Lie algebra g (it is called the holonomy Lie algebra) in $R^! = U(g)$ is the quotient of the free Lie algebra in 7 variables by the ideal generated by the 13 Lie commutators above. Now we have the formula

$$\frac{1}{R^!(z)} = \frac{1}{U(g)(z)} = \prod_{n=1}^{\infty}(1-t^n)^{\phi_i}$$

where the ϕ_i are the so-called LCS-ranks (the lower central series ranks). But these ranks can be calculated by a programme by Clas Löfwall [Lö3] which is called `liedim.m` and runs under Mathematica. It gives (in characteristic 0) the ranks $7, 8, 21, 42, 87, 105, 172, 264, 476, 816, 1516, 2704, 5068, 9312, 17484, \ldots$ but for the higher ranks you need the C-version of the programme unless your computer has lots of internal memory. Thus the Hilbert series $R^!$ can be calculated in degrees ≤ 15 and in these degrees it is described by the degree ≤ 15 part of following rather amazing formula:

$$\frac{1}{R^!(t)} = \frac{(1-2t)^8}{(1-t)^9}(1-t^3)^5(1-t^4)^{18}(1-t^5)^{39}(1-t^6)^{33}\prod_{n=4}^{\infty}(1-t^{2n-1})^{28}(1-t^{2n})^{24}$$

We indicate a possible proof below.

Second case (a "simplification-degeneration" of the Mac Lane arrangement): In the Mac Lane arrangement above ω being a primitive third root of unity satisfies $\omega^2 = -\omega - 1$ so if you replace ω^2 by $-\omega - 1$ in the definition of the Mac Lane arrangement above *and then* (this is brutal!) put $\omega = 1$ you obtain a new arrangement *mleas* in $\mathbf{C}^3$ defined by the polynomial with integer coefficients:

$$Q_{eas} = xyz(y-x)(z-x)(z+y)(z-2x+y)(z-x+2y) \tag{4.1}$$

The amazing thing now is that the corresponding hyperplane arrangement has an almost identical Orlik-Solomon algebra $R_{eas} \otimes E[z]$ but its homological properties are dramatically different: We have indeed that R_{eas} has all the relations of R *with the exeption of the relation x_3x_7 which is replaced by $x_3x_6x_7$*, so that $R = R_{eas}/(x_3x_7)$. and the Hilbert series $R_{eas}(t) = 1 + 7\,t + 14\,t^2$, which is close to $R(t) = 1 + 7\,t + 13\,t^2$. But now we have:

Theorem 4.1. The Koszul dual $R^!_{eas}$ of the hyperplane arrangement *mleas* has the following Hilbert series

$$\frac{1}{R^!_{eas}(t)} = \frac{(1-2\,t)^7}{(1-t)^7}\prod_{n=3}^{\infty}(1-t^n) \tag{4.2}$$

Corollary 4.2. The Ext-algebra of the algebra R_{eas} corresponding the hyperplane arrangement $mleas$ has a transcendental Hilbert series.

Proof of Corollary 4.2: The condition L_3 is satisfied since $m^3 = 0$.

SKETCH OF PROOF OF THEOREM 4.1 — We have $R^!_{eas} = R^!/([X_3, X_7])$. Furthermore $R^!_{eas}$ is the enveloping algebra of a Lie algebra g_{eas}. Now divide out g_{eas} with the Lie element of degree 3 $[X_6, [X_7, X_5]]$. We get an exact sequence of Lie algebras:

$$0 \longrightarrow ker \longrightarrow g_{eas} \longrightarrow g_{eas}/([X_6, [X_7, X_5]]) \longrightarrow 0 \tag{4.3}$$

where ker is defined by (4.3). I now claim that the Hilbert series of the enveloping algebra of $quot = g_{eas}/([X_6, [X_7, X_5]])$ is $(1-z)^7/(1-2\,z)^7$. Indeed the underlying Lie algebra has a basis of 7 elements $X_1, X_2, \ldots, X_7$ in degree 1. From now on we denote to simplify the elements $[X_i, [X_j, [X_k, \ldots]]]$ by $ijk \ldots$. With these notations we have the following basis of seven elements in degree 2:

42, 52, 53, 63, 64, 75, 76

These seven elements commute pairwise, and we have 14 elements of degree 3. One can prove that the Lie algebra decomposes in the sense of [Pa-Su], i.e. that the Lie algebra $quot$ in degrees ≥ 2 is a direct sum of 7 degree ≥ 2 parts of free Lie algebras in 2 variables. One can not use [Pa-Su] directly, but if S is R_{eas} without the relation $x_4x_5 - x_4x_7 + x_5x_7$ then S comes from the so-called X_2 arrangement which decomposes [Pa-Su] and has $S^!(z) = (1-z)^3/(1-2z)^5$ and the map $S \to R_{eas}$ can be analyzed as in section 3 above.

We now continue analyzing the kernel ker in (4.3). Clearly $675 = [X_6, [X_7, X_5]]$ is in this kernel and so are also the degree 4 elements $i675 = [X_i, [X_6, [X_7, X_5]]]$ for $i = 1 \ldots 7$. But we only get possibly non-zero elements (with different signs) for $i = 5$ and $i = 6$, (the last one can be written 6775) and in next degree we similarly get only one element 67775 etc. Therefore ker is $\leq$ 1-dimensional in each degree ≥ 3. Next we have the formulae in g_{eas}

$$\begin{gathered} -[X_5, 42] = [X_6, 42] = -[X_4, 52] = [X_6, 52] = [X_2, 53] = -[X_6, 53] = [X_1, 63] = \\ -[X_5, 63] = -[X_2, 64] = [X_5, 64] = -[X_1, 75] = [X_6, 75] = -[X_4, 76] = [X_5, 76] = 675 \end{gathered} \tag{4.4}$$

The other commutators lie in $quot$. Therefore if we define a graded vector space of dimension 1 in each degree ≥ 3 by $V_* = ke_3 \oplus ke_4 \ldots$ where the X_i operate as zero, with the exception of $X_5.e_n = e_{n+1}$ and $X_6.e_n = -e_{n+1}$, we get a $quot$-module V and by calculating

$H^2(quot, V)$ we find that there is a 2-cocycle $\gamma : quot \times quot \longrightarrow V$ on $quot$ with values in V which in lower degrees starts as in (4.4). Thus we have a Lie algebra g_γ defined by this cocycle which sits in the middle of an extension of Lie algebras where the kernel V is abelian:

$$0 \longrightarrow V \longrightarrow g_\gamma \longrightarrow quot \longrightarrow 0 \tag{4.5}$$

One now uses the Hochschild-Serre spectral sequence of the extension (4.5) to show that $Tor_{2,*}^{U(g_\gamma)}(k,k)$ is concentrated in degrees 2 . Now g_{eas} and g_γ are isomorphic (they have "the same" generators and relations). Thus ker is one-dimensional in each degree ≥ 3 This gives the formula (4.2) and Theorem 4.1 follows.

Remark 4.3. The previous reasoning with explicit cocycles is analogous to, but a little more complicated than Löfwall's and mine version of the Anick solution of the Serre-Kaplansky irrationality problem (cf. [Lö-Ro2], [Ro7,pages 454-456], [An]) as well as Lemaire's Bourbaki talk about these questions [Lem2]).

Now what about the Mac Lane arrangement ? The results are similar to those of the easier case just described. In fact the Lie algebra g_{eas} just studied has no center and the kernel Lie algebra ker of (4.3) is abelian. In order to do the similar reasoning for the Mac Lane (ML) arrangement one needs to divide the Lie algebra g_{ML} with the following *five* cubic Lie algebra elements:

$$[X_5,[X_7,X_3]],[X_6,[X_7,X_3]],[X_7,[X_6,X_3]],[X_7,[X_6,X_4]],[X_6,[X_7,X_5]] \tag{4.6}$$

leading again to a quotient Lie algebra $quot_{ML}$ whose enveloping algebra has Hilbert series $(1-z)^9/(1-2\,z)^8$ We still get an exact sequence of Lie algebras:

$$0 \longrightarrow ker_{ML} \longrightarrow g_{ML} \longrightarrow quot_{ML} \longrightarrow 0 \tag{4.7}$$

But now ker_{ML} is generated by the five elements (4.6) and is still situated in degrees ≥ 3 where its dimensions are

$$5, 18, 39, 33, 28, 24, 28, 24, 28, 24, 28, 24, 28, \ldots \tag{4.8}$$

and there is still a two-cocycle describing the extension (4.7). However now g_{ML} contains central elements, all situated in ker_{ML} and $[ker_{ML}, ker_{ML}]$ is contained in the center, which is 1-dimensional in degree 5, 9-dimensional in degree 6, and 4-dimensional in degrees $2n+1$ for $n \geq 3$ and zerodimensional in even degrees ≥ 8. If we divide out by the center

the Lie algebra $ker_{ML}/center$ is still situated in degrees ≥ 3 but its dimensions there are now

$$(4.9) \qquad 5, 18, 38, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, \ldots$$

Furthermore $ker_{ML}/center$ is abelian and $quot_{ML}$ operates on it in a similar but more complicated way than for g_{eas} above. But so far all this has only been proved in degrees ≤ 15.

Although the g_{eas} irrational case is much easier than the g_{ML} case it still uses a hyperplane arrangement with 8 hyperplanes leading to an algebra $R^!_{eas}$ in *seven variables*. One might still wonder it it would be possible to simplify further, i.e. get a hyperplane arrangement with seven or six hyperplanes and having an irrational series:

In [Ro4] we have in particular described all homological possibilities for the quotient of an exterior algebra in ≤ 5 variables by an ideal generated by ≤ 3 quadratic forms (the ring-theoretical classification was obtained in [E-K]). It is only in 5 variables that we can obtain non-finitely generated Ext-algebras (only one case, just studied above in section 1) or Ext-algebras with an irrational Hilbert series (three cases). These three cases are as follows (the numbering of cases is from [Ro4]) (in all these three cases the Hilbert series $R(t) = 1 + 5t + 7t^2$):
Case 12:

$$R_{12} = \frac{E(x,y,z,u,v)}{(xy, xz+yu+zv, uv)} \quad \text{with} \quad \frac{1}{R^!_{12}(t)} = (1-2t)^2 \prod_{n=1}^{\infty}(1-t^n)$$

Case 20:

$$R_{20} = \frac{E(x,y,z,u,v)}{(yz+xu, yu+xv, zu+yv)} \quad \text{with} \quad \frac{1}{R^!_{20}(t)} = \prod_{n=1}^{\infty}(1-t^{2n-1})^5(1-t^{2n})^3$$

Case 15:

$$R_{15} = \frac{E(x,y,z,u,v)}{(yz+xu, xv, zu+yv)} \quad \text{with} \quad \frac{1}{R^!_{15}(t)} = (1-2t) \prod_{n=1}^{\infty}(1-t^{2n-1})^3(1-t^{2n})^2$$

But we can not see how any of these algebras could come up from some hyperplane arrangement. If we study quotients of $E(x,y,z,u,v)$ with *four* quadratic forms there are still three other quotients (this time with Hilbert series $R(t) = 1 + 5t + 6t^2$) which *might* have irrational $R^!(t)$:

Case 21:

$$R_{21} = \frac{E(x,y,z,u,v)}{(yz+xu,yu+xv,zu+yv,uv)} \quad \text{with} \quad R^!_{21}(t) = 1+5t+19t^2+65t^3+211t^4+667t^5+$$
$$+2081t^6+6449t^7+19919t^8+61425t^9+189273t^{10}+583008t^{11}+1795509t^{12}+$$
$$+5529263t^{13}+17026752t^{14}+52431180t^{15}+161452384t^{16}+497162060t^{17}+$$
$$+1530914456t^{18}+4714152439t^{19}+14516309322t^{20}+44700127353t^{21}+$$
$$+137645268696t^{22}+423851580822t^{23}+\dots$$

Case 22:

$$R_{22} = \frac{E(x,y,z,u,v)}{(yz+xu,yu+xv,zu+yv,zv)} \quad \text{with} \quad R^!_{22}(t) = 1+5t+19t^2+65t^3+211t^4+666t^5+$$
$$+2071t^6+6387t^7+19609t^8+60054t^9+183672t^{10}+561340t^{11}+$$
$$+1714894t^{12}+5237883t^{13}+15996477t^{14}+\dots$$

Case 33:

$$R_{33} = \frac{E(x,y,t,u,v)}{(yz+xu,xv,zu+yv,uv)} \text{with}\, R^!_{33}(t) = 1+5t+19t^2+65t^3+212t^4+675t^5+2125t^6+$$
$$+6653t^7+\dots(21\,\text{terms}\,)+483131948638003t^{29}+1505474194810058t^{30}+\dots$$

but the following formula gives in this last case an indication about theta functions :

$$\frac{1}{(1-t)^2 R^!_{33}(t)} = 1-3t-t^2+t^3+2t^4+3t^5+t^6+$$
$$+t^7-t^8-t^9-2t^{10}-t^{11}-3t^{12}-t^{13}-t^{14}-t^{15}+t^{17}+$$
$$+t^{18}+2t^{19}+t^{20}+t^{21}+3t^{22}+t^{23}+t^{25}+t^{26}-t^{29}-t^{30}\dots$$

leading to the predictions that the coefficient for t^{31} should be -2 and that more precisely the series should continue as $-2t^{31}-t^{32}-t^{33}-t^{34}-3t^{35}+\dots$. But using the Backelin et al. programme BERGMAN [B] we have for the moment only been able to calculate the preceding series in degrees ≤ 30 and no precise theory is in sight. But I still do not know if these last three cases come from some hyperplane arrangements. In higher embedding dimensions (6,7,...) there are of course more irrational series and as we have indicated two of them in embedding dimension 7 come from complex hyperplane arrangements ...

Remark 4.4. The case $R^!_{20}$ (which comes from Jürgen Wisliceny and whose series was determined up to degree 67 by Czaba Schneider [Sch], Theorem 6.1) was completely determined in the super-Lie algebra case in [Lö-Ro1] (where we had a periodicity 4). Here

its treatment is easier (periodicity 2). Note that we have only described above what happens in characteristic 0. In case 20 we have different $R^!_{20}(z)$ in all characteristics and the same remark seems to be applicable to the cases 21,22,33.

5. Irrational or non finitely presented cases for other arrangements ?

In the sections above we have found two classes of unexpected complex hyperplane arrangements. An interesting question is to determine how rare those hyperplane arrangements are. The simplest arrangements are the so-called graphic arrangements: we have a simple graph Γ given with n vertices and t edges. The corresponding hyperplane arrangement $\mathcal{A}_\Gamma$ in $\mathbf{C}^n$ is defined by:

$$\mathcal{A}_\Gamma = \{x_i - x_j\} \text{ where } i < j, \text{ and where } (i, j) \text{ is an edge of } \Gamma$$

Such a graph leads as in section 1 to an Orlik-Solomon algebra which can be written in the form $R_\Gamma \otimes E(z)$ where R_Γ is a quotient of the exterior algebra in $t-1$ variables by homogeneous forms, and $E(z)$ is the exterior algebra in one variable z. It is therefore sufficient to analyze R_Γ. Now Lima-Filho and Schenk have recently proved [Li-Sch] that the Hilbert series of all $R^!_\Gamma$ are rational of a special form, and therefore it follows that the Hilbert series of the Ext-algebra of the Orlik-Solomon algebra of $\mathcal{A}_\Gamma$ is always rational, at least for those cases where the Hilbert series of R_Γ has the cube of its maximal ideal equal to 0 (and maybe in all cases, cf. remarks below). Indeed formula (2.17) above can be applied and gives an explicit rational formula. But non-finitely presented Ext-algebras can indeed occur for some graphs: In the book by about graphs by Harary [H] there is at the end an explit list of simple graphs with ≤ 6 vertices. We have gone through that list completely and we can report part of the results as Theorem 5.1 below (note that it is sufficient to analyze connected graphs, since the Orlik-Solomon algebra decomposes as a tensor product of the algebras corresponding to the connected components of the graph). It is known that the number of simple connected graphs with n vertices increases rapidly with n, according to the following table:

$n =$ Number of vertices of the graph:	2	3	4	5	6	7	8
Number of simple connected graphs with n vertices:	1	2	6	21	112	853	11117

We also use the numbering of the simple connected graphs from the home page of of Brendan McKay,

`http://cs.anu.edu.au/~bdm/data/graphs.html`

There you can download e.g. a file called graph4c.6g giving all the connected graphs on 4 vertices in so-called .g6-format, i.e. this file is in a strange compressed form. It looks like

CF

CU

CV

C]

C^

C~

Next you download the showg programme from the page

`http://cs.anu.edu.au/~bdm/data/formats.html`

and then e.g. the command

```
showg -eo1 graph4c.6g connected-graphs-of-order4
```

produces the file connected-graphs-of-order4 which looks as follows:

Graph 1, order 4.

4 3

1 4   2 4   3 4

Graph 2, order 4.

4 3

1 3   1 4   2 4

Graph 3, order 4.

4 4

1 3   1 4   2 4   3 4

Graph 4, order 4.

4 4

1 3   1 4   2 3   2 4

Graph 5, order 4.

4 5

1 3   1 4   2 3   2 4   3 4

Graph 6, order 4.

4 6

1 2   1 3   1 4   2 3   2 4   3 4

Thus we see that there are 6 connected graphs with 4 vertices. The first line after graph 4 only says that there are 4 vertices and 4 edges. The second line

1 3   1 4   2 3   2 4

says that these four edges are exactly those that connect vertices 1 and 3, vertices 1 and 4, vertices 2 and 3 and vertices 2 and 4, i.e. that the hyperplane arrangement is defined by

$$(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)$$

This is the only graphical arrangement for a graph with 4 vertices (the graph of a square) where the Orlik-Solomon algebra is not a Koszul algebra. The Orlik-Solomon algebra satisfies howver the condition L_3 (cf. Remark 2.3 in section 2) since $m^4 = 0$ and the Ext-algebra is finitely presented, and $Ext_R^*(k,k)$ has a rational Hilbert series.

For graphs of order 5 and 6 we have the following

THEOREM 5.1. a) Among the 21 connected graphs with 5 vertices, 15 give rize to Orlik-Solomon algebras (OS-algebras) that are Koszul. Among the 6 remaining non-Koszul algebras only *one* (corresponding to Graph 19 = the graph of a pyramid with a square basis) gives rise to a hyperplane arrangement where the Ext-algebra of the OS-algebra is not finitely presented, graphs 5,7,15,17,19 give OS-algebras that satisfies L_3, and graph 14 (the graph of a pentagon) has an OS-algebra that satisfies L_4, and since all $R(z)$ and $R^!(z)$ are rational, the Hilbert series of the 21 Ext-algebras are rational.

b) Among the 112 connected graphs with 6 vertices, 34 give rize to Orlik-Solomon algebras (OS-algebras) that are Koszul. Among the 78 remaining non-Koszul algebras only 7 (corresponding to graphs 71,74,100,102,107,108,109) have non-finitely presented Ext-algebras of their OS-algebras and one (corresponding to graph 98) has a finitely presented Ext-algebra, which however has an infinite number of relations between the relations. The condition L_5 is satisfied in one case (the graph of a 6-gon; more generally L_{n-1} is satisfied for the graph of an n-gon). The condition L_4 is satisfied for the graphs 48,95,98 and the condition L_3 is satisfied for the graphs
11,13,25,33,36,39,42,44,46,51,53,57,61,63,66,68,72,73,81,87,92,99,100,102,106,107,108,109.
There are 5 graphs for which no condition L_n is verified: 38,71,74,96,97. But also for these graphs the Hilbert series of the Ext-algebra can be analyzed and proved to be rational, so all these 112 graphs give rational series.

SKETCH OF PROOF OF PART OF THE THEOREM: a) The case of the graph 19 with 5 vertices gives rize to the arrangement defined by the polynomial

$$(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_5)(x_4 - x_5)$$

corresponding to a pyramid, where the vertex x_5 is at the top of the pyramid, and x_1, x_3, x_2, x_4 are at the basis. We have 8 factors and the OS-algebra is in 8 variables:

$$(5.1) \qquad \frac{E(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)}{((e_1-e_7)(e_3-e_7),(e_2-e_8)(e_3-e_8),(e_4-e_7)(e_6-e_7),(e_5-e_8)(e_6-e_8),(e_1-e_2)(e_2-e_5)(e_4-e_5))}$$

Let us now introduce new variables $x_i = e_i - e_5$ for $i \neq 5$ and $z = x_5$. Our algebra (5.1) becomes as earlier the tensor product of the exterior algebra $E(z)$ and the algebra in seven

variables (x_5 is missing !):

$$R = \frac{E(x_1,x_2,x_3,x_4,x_6,x_7,x_8)}{(x_1x_3 - x_1x_7 + x_3x_7, x_2x_3 - x_2x_8 + x_3x_8, x_4x_6 - x_4x_7 + x_6x_7, x_6x_8, x_1x_2x_4)}$$

It is now easy to see that the annihilator of x_6 in R is generated by x_6 and x_8 and and similarly that the annihilator of x_8 is generated by x_6 and x_8. Furthermore the intersection of the two ideals x_6 and x_8 is 0. Thus the ideal $a = (x_6, x_8)$ is a direct sum and $S = R/a$ has a linear resolution over R; more precisely we have $P_R^S(x,y) = 1/(1-2xy)$. Now apply a result by Rikard Bøgvad [Bø] which says that if $R \to S = R/a$ is an algebra map such that R/a has a linear R-resolution, then the map $R \to S$ is a large map in the sense of Levin [Le3]. In the proof of Lemma 2.3 b), page 4 in [Bø], there is a slight misprint on line 9 of the proof which should be:

" $\ldots \eta : R \to S$ is 1-linear, i.e. that $Tor_{i,j}^R(S,k) = 0$ if $i \neq j \ldots$ "

This has the consequence that the double series $P_R(x,y) = P_R^S(x,y)P_S(x,y)$ (i.e. the change of rings spectral sequence degenerates, cf. [Le3, Theorem 1.1, p. 209]. In our case $P_R^S(x,y) = 1/(1-2xy)$ and S now becomes the quotient of an exterior algebra in five variables:

$$S = \frac{E(x_1,x_2,x_3,x_4,x_7)}{(x_1x_3 - x_1x_7 + x_3x_7, x_4x_7, x_2x_3, x_1x_2x_4)}$$

But this algebra is essentially the algebra (1.2), treated in section 1 above. Indeed, let us first make the substitutions: $x_1 \longrightarrow x_1 + x_3,\ x_7 \longrightarrow x_7 + x_3$ and then interchange x_2 and x_3 and also interchange x_1 and x_7. We get the same algebra as in (1.2) in section 1; the only difference is that now the last variable is called x_7 (and not x_5 as in section 1). It follows that the double series of S is given by

$$\frac{1}{P_S(x,y)} = \frac{1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3}{1 - xy}$$

so that

$$\frac{1}{P_R(x,y)} = \frac{(1-2xy)(1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3)}{1 - xy} \tag{5.2}$$

From (5.2) we can now read off that $R(z) = 1 + 7z + 17z^2 + 14z^3 = (1+2z)(1+5z+14z^3)$ and that $R^!(z) = (1-z)/(1-2z)^4$ so that the formula L_3 holds true. One then proves that $gldim(R^!) = 4$ and that $Tor_{4,i}^{R^!}(k,k)$ is 1-dimensional for $i \geq 4$ and $Tor_{3,i}^{R^!}(k,k) = 0$ for $i \neq 3$ so that the algebra in section 1 which there needs an infinite number of generators now

comes back here as a subtle part of a graphic arrangement whose Ext-algebra is slightly better in that it is finitely generated but not finitely presented. Note in particular that $Ext^*_R(k,k)$ is *not* the tensor product of $Ext^*_S(k,k)$ and the free algebra on two variables of degree 1. The other non-Koszul cases in Theorem 5.1 a) are simpler and treated in a similar way. For graphs with 6 vertices (Theorem 5.1 b) similar procedures are used and the most complicated case is case 109, where the arrangement is defined by the polynomial:

$$(x_1-x_3)(x_1-x_4)(x_1-x_5)(x_1-x_6)(x_2-x_3)(x_2-x_4)(x_2-x_5)(x_2-x_6)(x_3-x_5)(x_3-x_6)(x_4-x_5)(x_4-x_6)$$

where we still have a non-finitely presented Ext-algebra. But the most interesting pairs of examples are 107 and 74 which have the same Hilbert series *both* for the Orlik-Solomon algebra *and* for the quadratic dual $R^!$ and the first algebra satisfies L_3 and the second does not. But they have both Ext-algebras that are not finitely presented. In particular the Tor (or Ext) of the Orlik-Solomon algebras differ, the first difference occurs for $Tor^{\mathcal{OS}}_{4,6}(k,k)$ which has dimension 9 for the case 107 and dimension 10 for case 74. A similar phenomenon occurs for the cases 87 and 71 where the first $Tor^{\mathcal{OS}}_{4,6}(k,k)$ has dimension 16 for the case 87 (but here the Ext-algebra is finitely presented) and the second case 71 has $Tor^{\mathcal{OS}}_{4,6}(k,k)$ of dimension 17 and the Ext-algebra is not finitely presented in that case.

Remark 5.2. I have also studied many of the 853 cases corresponding to graphs with seven vertices. Everything in sight leads to rational Hilbert series for the Ext-algebras.

Remark 5.3. When I lectured about this at Stockholm University, Jörgen Backelin made an interesting observation:

1) For graphds with 5 vertices the only case of non-finitely presented Ext-algebras comes from the case when you remove two disjoints edges from the complete graph on 5 vertices (case 19).

2) If you remove three disjoint edges in the complete graph on 6 vertices you get the case 109 which is the most complicated one for graphs of order 6.

3) This leads to a conjecture that if you remove $[n/2]$ disjoint edges from the complete graph on n vertices ($n \geq 7$) then you should get a very interesting situation.

Remark 5.4. Here is another example: The Example 1.3 of [Fi-Sch] where G is "the one skeleton of the Egyptian pyramid and the one skeleton of a tetrahedron sharing a single triangle" we have $1/R^!(t) = (1-2t)^4(1-3t)$ and $R(t) = 1+11t+48t^2+103t^3+107t^4+42t^5$ leading to the formula

$$\frac{1}{P_R(x,y)} = 1 - 11\,x^2y^2 + 48\,x^3y^3 - x^2y^3 - 104\,x^3y^3 + 5\,x^3y^4 + 112\,x^4y^4 - 6\,x^4y^5 - 48\,x^5y^5$$

i.e. this is another one of the cases where (2.17) is true but $m^3 \neq 0$.

Furthermore, the global dimension of $R^!$ is 5 and the Ext-algebra is finitely generated but *not* finitely presented.

Remark 5.5. The preceding results (which should satisfy the request of one of the referees) show that the behaviour of the graphic arrangements in the non-Koszul case are rather unpredictable (but the irrational case is rare and it is quite probable that it does not occur for graphic arrangements).

6. Questions of Milnor, Grigorchuk, Zelmanov, de la Harpe and irrationality.

In section 4 we have presented two hyperplane arrangements, the Mac Lane arrangement ML and an easier variant $mleas$ =(4.1) which both have irrational Hilbert series for the corresponding $R^!$. In section 4 we also presented the three possibilities for irrational Hilbert series for $R^!$ when R is an arbitrary quotient of an exterior algebras in five variables with an ideal generated by three quadratic forms (the only possibilities): Cases R_{12}, R_{20} and R_{15}. It is seems difficult to achieve similar examples for hyperplane arrangements using five variables. But $mleas$ can be considered as a higher variant of R_{12} and similarly ML can be considered as a higher variant of R_{15} (indeed in this last case there are central elements in odd degrees ≥ 3 , so that we get the exponents 2,2,2,... in the infinite product formula for case $R^!_{15}(t)$ in section 4 when we have divided out the center. But so far I have not found any hyperplane arrangement corresponding (or similar) to the irrational case R_{20} in section 4. If such a hyperplane arrangement existed it would in particular lead to results about growth of groups and groups of finite width. Let me be more precise: First recall that if $\mathcal{A}$ is *any* finite complex hyperplane arrangement in $\mathbf{C}^n$ and if $G = G(\mathcal{A})$ is the fundamental group of the complement of the union of the corresponding hyperplanes in $\mathbf{C}^n$ then G is finitely presented — indeed the complement has the homotopy type of a finite CW-complex ([Or], Proposition 5.1, p. 43). Let

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$$

be the descending lower central series of G defined inductively by $G_1 = G$ and $G_k = [G_{k-1}, G_1]$ (for $k \geq 2$). We have a structure of a graded Lie-ring (which can have torsion):

$$gr(G) = \bigoplus_{i \geq 1} \frac{G_i}{G_{i+1}}$$

where the graded Lie structure is defined as follows: Let $\bar{x}$ and $\bar{y}$ be elements in G_i/G_{i+1} and G_j/G_{j+1} respectively,and let them be represented by x and y in G_i and G_j. Then $xyx^{-1}y^{-1}$ lies in G_{i+j} and its image in G_{i+j}/G_{i+j+1} is denoted by $[\bar{x}, \bar{y}]$. It was proved

by Kohno [K] that we have an isomomorphism of graded Lie algebras

$$gr(G) \otimes_Z \mathbf{Q} \simeq \bigoplus \eta^i \tag{6.1}$$

where η is the Lie algebra of primitive elements in the subalgebra generated by $Ext^1_{OS(\mathcal{A})}(Q,Q)$ of the Yoneda Ext-algebra of the Orlik-Solomon algebra of $\mathcal{A}$. Now G_{ML} and is G_{mleas} are finitely presented groups. Therefore, if we could find a hyperplane arrangement (probably in high embedding dimension) corresponding to the case R_{20} or similar, we would have at the same time found a *finitely presented group* G such that the Lie algebra $gr(G) \otimes_Z \mathbf{Q}$ is infinite and of finite width (i.e the dimensions of the η^i in (6.1) are bounded (for further terminology and results I refer to the surveys [delH],[Ba-Gr], [Gr-P] and the literature cited there). If so one would probably be close to *finitely presented* groups having intermediate growth. Note that it is not expected that such groups exist (cf. Conjecture 11.3 in [Gr-P], where one states two lines earlier that the existence of such groups is a major open problem in the field, and research problem VI.63 on page 297 of [delH]). But our two groups corresponding to the Mac Lane arrangement ML and its easier variant $mleas$ give at least *finitely presented groups* with irrational growth series (Hilbert series) of $U(gr(G) \otimes_Z \mathbf{Q})$.

7. Final remarks.

It is interesting to note that 32-33 years ago Jean-Michel Lemaire was in his thesis [Lem1] inspired by the Stallings group-theoretical example [St] (now used again in [Di-Pa-Su] !) to construct a finite simply-connected CW-complex X such that the homology algebra of the loop space $H_*(\Omega X, \mathbf{Q})$ was not finitely presented (not even finitely generated). In [Ro1] I used a general recipe which in particular could be used to translate Lemaire's results to local commutative ring theory to obtain a local ring (R, m) such that the Yoneda Ext-algebra $\mathrm{Ext}^*_R(k,k)$ was not finitely generated, thereby solving in the negative a problem by Gerson Levin [Le1]. The example in section 1 above is just a skew-commutative variant of my example in [Ro1], but with a quick direct proof, which hopefully should satisfy mathematicians working with arrangements of hyperplanes. The theory of section 2 above, combined with more difficult variants of the later developments in the 1980:s about a question of Serre-Kaplansky [Lem2] are here shown to be useful for solving the second problem of Denham-Suciu [De-Su].

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